

Image Reduction With Local Reduction Operators

Daniel Paternain, Humberto Bustince, Javier Fernandez, Gleb Beliakov and Radko Mesiar

Abstract—In this work we propose an image reduction algorithm based on weak local reduction operators. We use several averaging functions to build these operators and we analyze their properties. We present experimental results where we apply the algorithm and weak local reduction operators in procedures of reduction, and later, reconstruction of images. We analyze these results over natural images and noisy images.

I. INTRODUCTION

Image reduction consists in diminishing the resolution or dimension of the image while keeping as much information as possible. Image reduction can be used to accelerate computations on an image ([13], [8]), or just to reduce its storage cost ([10]).

In the literature there exist many methods for image reduction. We can divide these methods in two groups. In the first group, the image is separated in blocks. Then each block is treated in an independent way. The reduced image is made by composition of the results of the algorithm in each block ([7], [9]). For the second group, on the contrary, the image is considered in a global way ([10]). For this work we focus on the first group of algorithms. Working with small pieces of the image allows to design simple reduction algorithms. Besides, as the algorithm act locally on the image, we can develop reduction algorithms with better features as keeping some properties of the image (i.e. edges) or reducing some type of noise.

The problem we consider is the following: To build an image reduction algorithm such that, for each block in the image, we obtain a single value that represents all the elements in that block, and hence, such that we keep as much information as possible.

We propose the concept of weak local reduction operator that takes a block of the image and returns a single point satisfying certain conditions. We are going to use averaging functions for building weak local reduction operators, since these particular aggregation functions have been widely studied ([1], [4], [6]). Moreover, we are going to analyze the properties of these averaging functions and how they affect to image reduction.

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There is no exact way of determining the best reduction method. It depends on a particular application we are considering. In this work, to decide whether one reduction is better than another, we reconstruct the original image from the reduction using the bilinear interpolation of MATLAB. We chose this reconstruction method since we also implement our methods with MATLAB. We analyze how weak local reduction operators operate in the reduction of images with different types of noise.

The remainder of the work is organized as follows. In Section 2 we briefly introduce some theoretical concepts. In Section 3 we present the definition of weak local reduction operators. In Section 4 we present our image reduction algorithm. In Section 5 we build weak local reduction operators. Finally, we show some experimental results, as well as some brief conclusions and future lines of research.

II. PRELIMINARIES

We start by recalling some concepts that will be used along this work.

Definition 1: An aggregation function of dimension n (n -ary aggregation function) is a non-decreasing mapping $M : [0, 1]^n \rightarrow [0, 1]$ such that $M(0, \dots, 0) = 0$ and $M(1, \dots, 1) = 1$.

Definition 2: Let $M : [0, 1]^n \rightarrow [0, 1]$ be a n -ary aggregation function.

- (i) M is said to be idempotent if $M(x, \dots, x) = x$ for any $x \in [0, 1]$.
- (ii) M is said to be homogeneous if $M(\lambda x_1, \dots, \lambda x_n) = \lambda M(x_1, \dots, x_n)$ for any $\lambda \in [0, 1]$ and for any $(x_1, \dots, x_n) \in [0, 1]^n$.
- (iii) M is said to be shift-invariant if $M(x_1 + r, \dots, x_n + r) = M(x_1, \dots, x_n) + r$ for all $r > 0$ such that $0 \leq x_i + r \leq 1$ for any $i = 1, \dots, n$.

A complete characterization for shift-invariance and homogeneity of aggregation functions can be found in [11], [12].

We know that a triangular norm (t-norm for short) $T : [0, 1]^2 \rightarrow [0, 1]$ is an associative, commutative, non-decreasing function such that $T(1, x) = x$ for all $x \in [0, 1]$. A basic t-norm is the minimum ($T_M(x, y) = \wedge(x, y)$). Analogously, a triangular conorm (t-conorm for short) $S : [0, 1]^2 \rightarrow [0, 1]$ is an associative, commutative, non-decreasing function such that $S(0, x) = x$ for all $x \in [0, 1]$. A basic t-conorm is the maximum ($S_M(x, y) = \vee(x, y)$).

III. LOCAL REDUCTION OPERATORS

In this work, we consider an image of $n \times m$ pixels as a set of $n \times m$ elements arranged in rows and columns. Hence we consider an image as a $n \times m$ matrix. Each element of the matrix has a value in $[0, 1]$ that will be calculated by

normalizing the intensity of the corresponding pixel in the image. We use the following notation.

- $\mathcal{M}_{n \times m}$ is the set of all matrices of dimension $n \times m$ over $[0, 1]$.
- Each element of a matrix $A \in \mathcal{M}_{n \times m}$ is denoted by a_{ij} with $i \in \{1, \dots, n\}$, $j \in \{1, \dots, m\}$.
- Let $A, B \in \mathcal{M}_{n \times m}$. We say that $A \leq B$ if for all $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$ the inequality $a_{ij} \leq b_{ij}$ holds.
- Let $A \in \mathcal{M}_{n \times m}$ and $c \in [0, 1]$. $A = c$ denotes that $a_{ij} = c$ for all $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$. In this case, we will say that A is constant matrix or a flat image.

Our objective is to reduce images acting on blocks of the image. We propose the definition of weak local reduction operators as operators that take a block of the image and return a single value. We impose two properties that, in our opinion, the operators must fulfill: monotonicity and idempotence.

Definition 3: A weak local reduction operator WO_{RL} is a mapping $WO_{RL} : \mathcal{M}_{n \times m} \rightarrow [0, 1]$ that satisfies

- (WORL1) For all $A, B \in \mathcal{M}_{n \times m}$, if $A \leq B$, then $WO_{RL}(A) \leq WO_{RL}(B)$.
- (WORL2) If $A = c$ then $WO_{RL}(A) = c$.

Remark: We call our operators weak local reduction operators since we demand the minimum number of properties that, in our opinion, a local reduction operator must fulfill.

Definition 4: We say that a weak reduction operator WO_{RL} is:

- (WORL3) homogeneous if $WO_{RL}(\lambda A) = \lambda \cdot WO_{RL}(A)$ for all $A \in \mathcal{M}_{n \times m}$ and $\lambda \in [0, 1]$
- (WORL4) stable under translation (shift-invariant) if $WO_{RL}(A + r) = WO_{RL}(A) + r$ for all $A \in \mathcal{M}_{n \times m}$ and $r \in [0, 1]$ such that $0 \leq a_{ij} + r \leq 1$ whenever $i \in \{1, \dots, n\}, j \in \{1, \dots, m\}$

IV. IMAGE REDUCTION ALGORITHM

Given an image $A \in \mathcal{M}_{n \times m}$ and a reduction block size $n' \times m'$ (with $n' \leq n$ and $m' \leq m$), we propose the following algorithm:

- (1) Choose a weak local reduction operator.
- (2) Divide the image A into disjoint blocks of dimension $n' \times m'$.
If n is not a multiple of n' or m is not a multiple of m' we suppress the smallest number of rows and/or columns in A that ensures that these conditions hold.
- (3) Apply the weak local reduction operator to each block.

V. CONSTRUCTION OF WEAK LOCAL REDUCTION OPERATORS FROM AVERAGING FUNCTIONS

In this section we study construction methods of weak local reduction operators using averaging functions. We also study the properties of some families of averaging functions.

Proposition 1: Let M be an idempotent aggregation function. The operator defined by

$$WO_{RL}(A) = M(a_{11}, a_{12}, \dots, a_{1m}, \dots, a_{n1}, \dots, a_{nm})$$

for all $A \in \mathcal{M}_{n \times m}$ is a weak local reduction operator.

Example 1: a) Take $M = T_M$. In Figure 1 we apply the weak local reduction operator obtained from T_M to the image (a) and we obtain image (a1).

b) Take $M = S_M$. In the same figure, we apply the weak local reduction operator obtained from S_M to image (a) and we obtain image (a2).



(a)



(a1)



(a2)

Fig. 1. Reduction of Cameraman using minimum and maximum and block size of 2×2

In images (a) and (b) of Figure 2 we add some salt and pepper noise to the original Cameraman image. Image (a) has noise level of 0.05 (i.e., around 5% of pixels are affected by noise). Image (b) has noise level of 0.1. Applying the same procedure than in the previous figure:

- a) We apply the weak local reduction operator obtained from T_M to the images (a) and (b) and we obtain images (a1) and (b1).
- b) We apply the weak local reduction operator obtained from S_M to images (a) and (b) and we obtain images (a2) and (b2).

Observe that these two operators minimum and maximum are not good local reduction operators. If we take the minimum over a block with noise we always obtain the value 0. Analogously, if we consider the maximum and apply it to a block with noise, we always recover the value 1. In this way we lose all information about the elements in the block

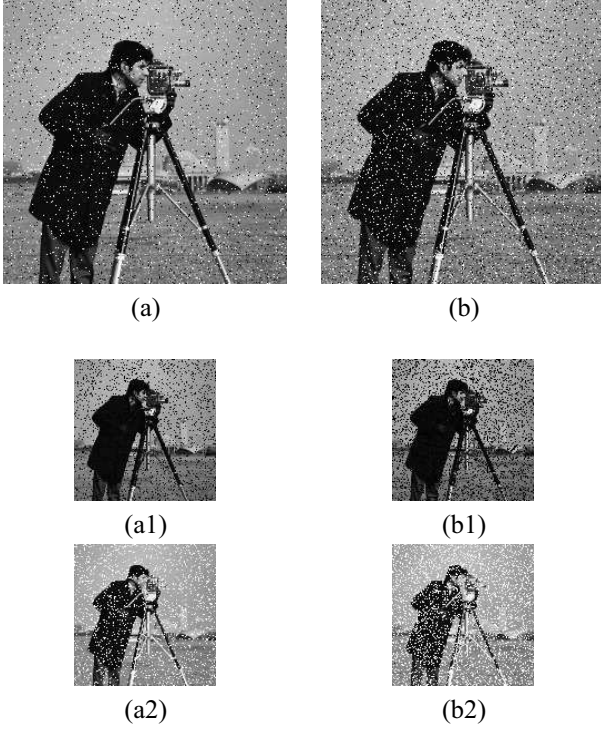


Fig. 2. Reduction of Cameraman with noise using minimum and maximum and block size 2×2

that have not been affected by noise. This behavior can be seen in Figure 2. Moreover, The greater the level of the noise is, the worse the quality of the reduced image. This fact leads us to study other aggregation functions.

Proposition 2: The following items hold:

- (1) $WO_{RL}(A) = T_M(a_{11}, a_{12}, \dots, a_{1m}, \dots, a_{n1}, \dots, a_{nm})$ is a weak local reduction operator that verifies (WORL3) and (WORL4).
- (2) $WO_{RL}(A) = S_M(a_{11}, a_{12}, \dots, a_{1m}, \dots, a_{n1}, \dots, a_{nm})$ is a weak local reduction operator that verifies (WORL3) and (WORL4).

Proof: It follows from the fact that both weak local reductions operators constructed from minimum and maximum satisfies (WORL3) and (WORL4).

A. Weighted quasi arithmetic means

Definition 5: Let $g : [0, 1] \rightarrow [-\infty, \infty]$ be a continuous and strictly monotone function and $w = (w_1, \dots, w_n)$ a weighting vector such that $\sum_{i=1}^n w_i = 1$. A weighted quasi-arithmetic mean is a mapping $M_g : [0, 1]^n \rightarrow [0, 1]$ defined as

$$M_g(x_1, \dots, x_n) = g^{-1} \left(\sum_{i=1}^n w_i g(x_i) \right)$$

Proposition 3: Let $M_g : [0, 1]^n \rightarrow [0, 1]$ be a weighted quasi-arithmetic mean. The operator defined as

$$WO_{RL}(A) = g^{-1} \left(\sum_{i=1}^n \sum_{j=1}^m w_{ij} g(a_{ij}) \right)$$

for all $A \in \mathcal{M}_{n \times m}$ is a weak local reduction operator.

Notice that from Definition 5 we can generate well-known aggregation functions as, for instance, the weighted arithmetic mean ($g(x) = x$) and the weighted harmonic mean ($g(x) = x^{-1}$).

In Figure 3 we apply the following weak local reduction operators:

- a) We apply the weak local reduction operator constructed from arithmetic mean to image (a) and we obtain image (a1).
- b) We apply the weak local reduction operator constructed from harmonic mean to image (a) and we obtain image (a2).

In Figure 4 we have added some salt and pepper noise to the House image. In image (a) we have added noise with a level of 0.05, whereas in image (b), noise with a level of 0.1. Following the same procedure:

- a) We apply the weak local reduction operator constructed from arithmetic mean to image (a) and we obtain images (a1) and (b1).
- b) We apply the weak local reduction operator constructed from harmonic mean to image (a) and we obtain images (a2) and (b2).

Notice that the two operators do not react in the same to this kind of noise. If we take the arithmetic mean, the image that we obtain is less affected than if we use the harmonic mean. This is due to the fact that if we apply the harmonic mean over a block with noise, we always get the value 0.

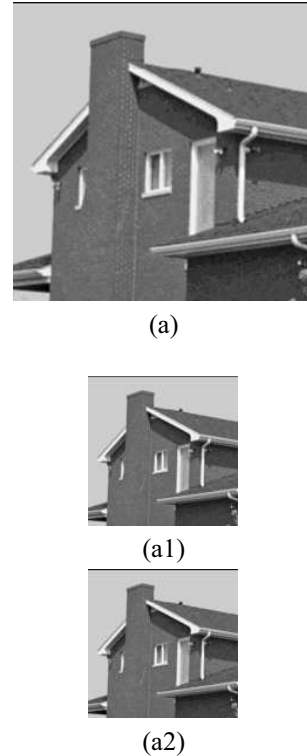


Fig. 3. Reduction of House using arithmetic mean and harmonic mean and block size of 2×2

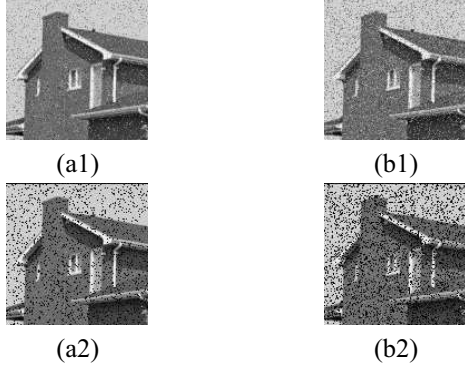


Fig. 4. Reduction of House with noise using arithmetic mean and harmonic mean and block size of 2×2

These results have led us to study properties (*WORL3*) and (*WORL4*) in weak local reduction operators built from weighted quasi-arithmetic means.

Proposition 4: A weak local reduction operator built from a weighted quasi-arithmetic with $w_{ij} = \frac{1}{n \cdot m}$ satisfies (*WORL3*) if and only if

$$WO_{RL}(A) = \left(\prod_{i=1}^n \prod_{j=1}^m a_{ij} \right)^{\frac{1}{n \cdot m}} \quad \text{or}$$

$$WO_{RL}(A) = \left(\sum_{i=1}^n \sum_{j=1}^m \frac{a_{ij}^\alpha}{n \cdot m} \right)^{\frac{1}{\alpha}} \quad \text{with } \alpha \neq 0$$

for all $A \in \mathcal{M}_{n \times m}$.

Proof: See page 118 of [6].

In Figure 5 we illustrate property (*WORL3*). Image (a) is Peppers image with random noise (white pixels). Image (b) has been obtained multiplying the intensity of each of the pixels of (a) by $\lambda = 0.5$. That is,

$$(b) = 0.5 \cdot (a).$$

Under these conditions, we consider the following weak local reduction operators.

- the harmonic mean in the second row.
- the following quasi arithmetic mean

$$M_g(x_1, \dots, x_n) = \begin{cases} \frac{\sqrt[n]{\prod_{i=1}^n x_i}}{\sqrt[n]{\prod_{i=1}^n x_i} + \sqrt[n]{\prod_{i=1}^n (1-x_i)}} & \text{if } \{0, 1\} \not\subseteq \{x_1, \dots, x_n\} \\ 0 & \text{otherwise} \end{cases}$$

in the third row

We see that

$$(b1) = 0.5 \cdot (a1),$$

so they keep the same proportion as images (a) and (b). However, it is visually clear that

$$(b2) \neq 0.5 \cdot (a2).$$

This is due to the fact that the second aggregation function that we have used does not satisfy (*WORL3*).

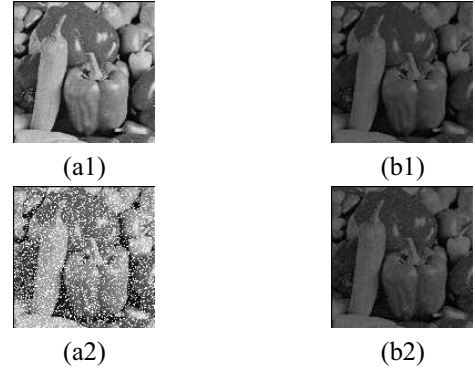
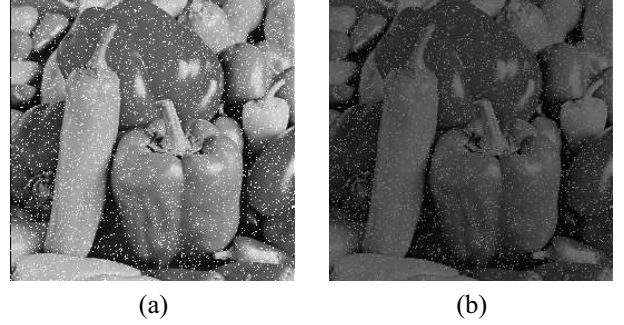


Fig. 5. Test of property (*WORL3*) of weak local reduction operators

Proposition 5: A weak local reduction operator built from a weighted quasi-arithmetic mean with $w_{ij} = \frac{1}{n \cdot m}$ satisfies (*WORL4*) if and only if

$$WO_{RL}(A) = \frac{1}{n \cdot m} \sum_{i=1}^n \sum_{j=1}^m a_{ij} \quad \text{or}$$

$$WO_{RL}(A) = \frac{1}{\alpha} \log \left(\sum_{i=1}^n \sum_{j=1}^m \frac{e^{\alpha a_{ij}}}{n \cdot m} \right) \quad \text{with } \alpha \neq 0$$

for all $A \in \mathcal{M}_{n \times m}$

Proof: See page 118 in [6].

In Figure 6 we illustrate property (*WORL4*). The normalized intensity of the pixels in image (a) vary from 0 to 0.5. Image (b) corresponds to add $r = 0.5$ to each of the intensities of the pixels in image (a). That is,

$$(b) = (a) + 0.5.$$

We apply the following weak local reduction operators:

- the arithmetic mean in the second row

- the harmonic mean in the third row.

Observe that

$$(b1) = (a1) + 0.5.$$

However, it is visually clear that

$$(b2) \neq (a2) + 0.5.$$

This is due to the fact that the arithmetic mean satisfies (*WORL4*) whereas the harmonic mean does not.

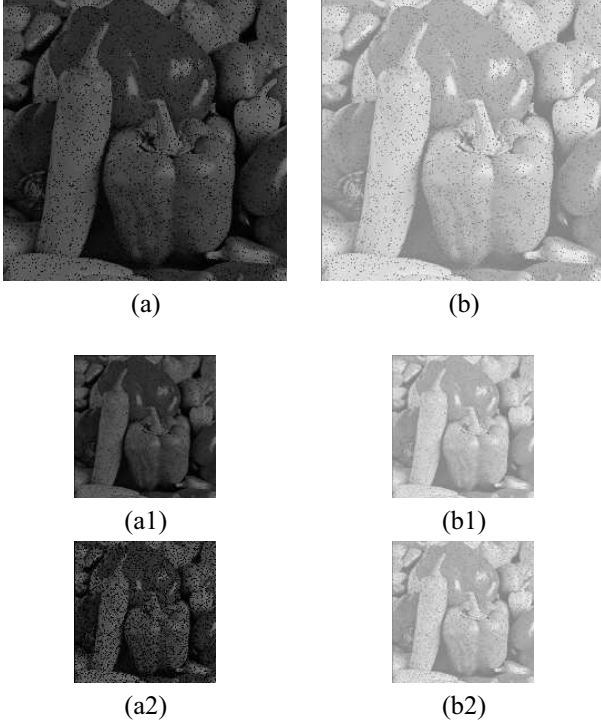


Fig. 6. Test of property (*WORL4*) of weak local reduction operators

B. Median

Proposition 6: The operator defined as

$$WO_{RL}(A) = Med(a_{11}, \dots, a_{1m}, \dots, a_{n1}, \dots, a_{nm})$$

for all $A \in \mathcal{M}_{n \times m}$, where *Med* denotes the median, is a weak local reduction operator verifying (*WORL3*) and (*WORL4*).

Proof: It is straightforward.

In Figure 7 we show the original Lena image (image (a)). We take as weak local reduction operator the one defined from the median and obtain image (a1).

In Figure 8 we add salt and pepper noise to Lena image with a level of 0.05 to get image (a) and with a level of 0.1 to get image (b). We apply the same weak local reduction operator constructed from the median and obtain (a1) and (b1). Observe that for this kind of reduction operators, noise does not have as much influence as for others. The reason is that the median is not sensitive to the magnitude of the extreme values.

Remark: Observe that we can build weak local reduction operators based on Choquet integrals. In particular, if we

impose symmetry, we get OWA operators and the median as prominent cases.



Fig. 7. Reduction of image Lena using the median operator with block sizes of 2×2 .



Fig. 8. Reduction of image Lena with noise using the median operator with block sizes of 2×2 .

VI. EXPERIMENTAL RESULTS

To settle which the best reduction is, we are going to reconstruct the reduced images in order to make a comparison with the original one. As we have already said in the Introduction, there is no single method of determining which the best reduction is. In this work, we have reconstructed the reduced images using the bilinear interpolation provided by MATLAB.

There exist many methods to calculate the similarity between the original image and the reconstructed one. In fact, we know that there is a relation between the different types

of errors (absolute mean error, quadratic mean error, etc.) and the aggregation function to be considered in each case ([4]). We will study in the future the relationship between the error measure and the aggregation function in image comparison. On the other hand, we can consider an image as a fuzzy set ([3]). For this reason, we are going to use fuzzy image comparison indexes. In [2] an in depth study of such indexes is carried out. In our work we are going to consider the Similarity measure based on contrast de-enhancement. This index has been used, for instance, in [5] and it satisfies the six properties demanded in [2] to similarity indexes: reflexivity, symmetry, reaction to binary images, the comparison between two images must be the same as the comparison between their negatives, the comparison between an image and its negative indicate how far away from binarity the image is and reaction to the noise. With the notations we are using, given $A, B \in \mathcal{M}_{n \times m}$, this index is given by:

$$S(A, B) = \frac{1}{n \cdot m} \sum_{i=1}^n \sum_{j=1}^m 1 - |a_{ij} - b_{ij}|.$$

In these experiments, we are going to use weak local reduction operators based on five averaging functions: minimum, harmonic mean, arithmetic mean, median and maximum. We are going to reduce and reconstruct the images in Figure 9. Observe that we take reduction blocks of size 2×2 . If the original images are of dimension 256×256 , the reduced dimensions are 128×128 .

In Table I we show the result of the comparison by means of the S index between the original images in Figure 9 and the reconstructed ones.

Observe that, in general, we obtain very good results. On average, the best result for the four images is obtained with the median. Results are very similar if we take the arithmetic mean or the harmonic mean. Results are worse if we take minimum or maximum. This is due to the fact that with these two operators we only take into account a single value for each block, which needs not to be representative of the rest of values in that block.

TABLE I
COMPARISON BETWEEN RECONSTRUCTED AND ORIGINAL IMAGES

Av. Fun.	Image (a)	Image (b)	Image (c)	Image (d)
Minimum	0,9617	0,9595	0,9448	0,97
Harm. Mean	0,9733	0,9713	0,9606	0,9789
Arith. Mean	0,9733	0,9713	0,9607	0,9775
Median	0,9741	0,9719	0,9611	0,9778
Maximum	0,9629	0,96	0,9448	0,9666

To analyze the reaction to noise of weak local reduction operators, we have added salt and pepper noise to original images with level of 0.05, and we have obtained the images in Figure 10. In Table II we show the comparison between the reconstructions and the original images. In these conditions, the best result is also obtained using the median as weak local reduction operator. This is due to the fact that the value provided by the median is not affected by salt and pepper

noise. Moreover, we observe that the operators given by the minimum, the harmonic mean and the maximum are very sensitive to this noise. For the first two ones, a single pixel of 0 intensity determines that the value for the corresponding block is also 0. For the maximum, if there is a pixel with intensity equal to 1, then the result is also equal to 1.

TABLE II
COMPARISON BETWEEN RECONSTRUCTED AND ORIGINAL IMAGES WITH NOISE (P=0.05)

Av. Fun.	Image (a)	Image (b)	Image (c)	Image (d)
Minimum	0,9001	0,9168	0,9091	0,9332
Harm. Mean	0,9125	0,9286	0,9254	0,9409
Arith. Mean	0,958	0,9576	0,9492	0,9634
Median	0,9724	0,9701	0,9596	0,9756
Maximum	0,9393	0,9174	0,8967	0,9162

In Figure III we have added more salt and pepper noise to the images, up to a noise level of 0.1. In Table II we show a comparison between the reconstructed and the original images. The results are then more pronounced: weak local reduction operators constructed from minimum, harmonic mean and maximum give very bad results; the weak local reduction operator constructed from the arithmetic mean worsen its results in a less acute way; finally, the operator constructed from the median keeps its good results.

TABLE III
COMPARISON BETWEEN RECONSTRUCTED AND ORIGINAL IMAGES WITH NOISE (P=0.1)

Av. Fun.	Image (a)	Image (b)	Image (c)	Image (d)
Minimum	0,8419	0,8727	0,8773	0,897
Harm. Mean	0,856	0,8857	0,8947	0,9054
Arith. Mean	0,9435	0,9446	0,9388	0,9495
Median	0,9676	0,9655	0,9555	0,9707
Maximum	0,9152	0,8747	0,8514	0,866

A comparison between results is shown in Figure 12. For each of the weak local reduction operators we show the average of the S index in the four test images. The first column of each operator corresponds to reduction and reconstruction of the original images. The second column corresponds to images with salt and pepper noise and $p = 0.05$. The third column corresponds to images with salt and pepper noise and $p=0.1$.

VII. CONCLUSIONS

In this work we have axiomatically defined local reduction operators. We have studied how to construct these operators by means of averaging functions. We have analyzed which properties are satisfied by some of these aggregation-based reduction operators.

From our operators, we have proposed an image reduction algorithm. To settle which is the best local reduction operator, we have proposed an application based on reconstructing the original images from the reduced ones. To compare images we have used a fuzzy similarity index. We have seen that,



Fig. 9. Original images for experimental results

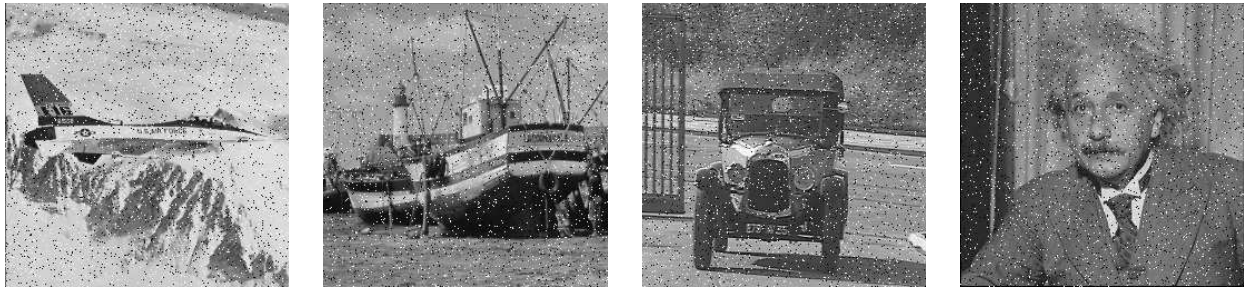


Fig. 10. Original images with noise ($p=0.05$) for experimental results

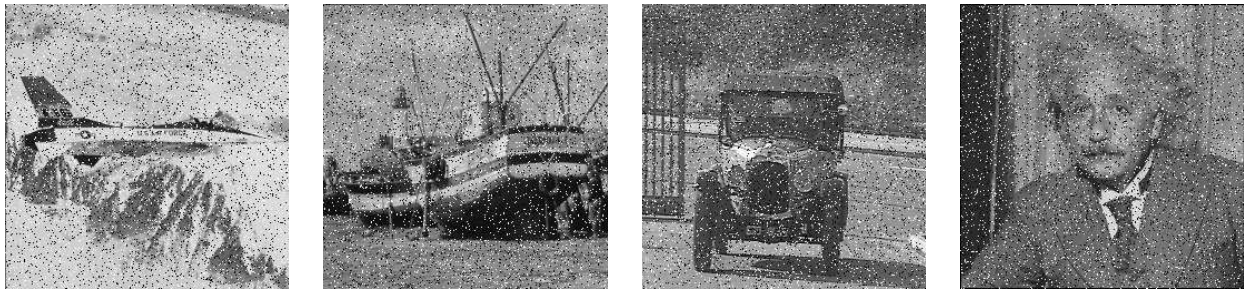


Fig. 11. Images with noise ($p=0.1$) for experimental results

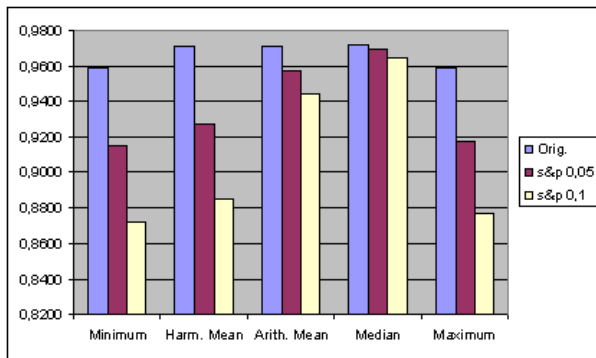


Fig. 12. Comparison of reconstructions with different weak local reduction operators and different noise

in all of the cases, the best weak local reduction operator is provided by the median. Moreover, this operator is not affected by salt and pepper noise.

ACKNOWLEDGMENT

This work has been partially supported by grants TIN2007-65981, VEGA 1/0080/10 and APVV-0012-07.

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